Journal of Engineering Mathematics 23: 29–51, 1989

Non-axisymmetric instability of polar orthotropic annular plates

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Received 29 March 1988; accepted 6 June 1988

Abstract. The non-axisymmetric instability of polar orthotropic annular plates under inplane uniform radial pressures is studied by use of the shooting method. The characteristic equations and eigenvalues under a variety of edge conditions are given. Under two appropriate hypotheses, we prove that all eigenvalues are bifurcation points. Hence, it is possible that non-axisymmetric buckled and post-buckled states branch from axisymmetric unbuckled states of an annular plate. Asymptotic formulae for buckled states are obtained and curves for the deflection and stress are shown.

1. Introduction

In this paper, we investigate the buckling and post-buckling behaviour of polar orthotropic annular plates. We take the governing equations to be those von Kármán's equations generalized by the authors in [6, 7], expressed in polar coordinates (r, θ). These form a system of fourth-order coupled partial differential equations with variable coefficients which contain several geometrical and material parameters. Critical loads for the non-axisymmetric instability of annular plates with a variety of boundary conditions and material parameters have been computed using variational and finite-difference methods [1–5], but non-axisymmetric buckling and post-buckling behaviour was not discussed in those papers. In this paper, we employ the shooting method [9] to investigate the non-axisymmetric buckled and post-buckled states of polar orthotropic annular plates under a variety of boundary conditions and subjected to inplane uniform radial pressures. The characteristic equations and their eigenvalues are obtained. We show that all eigenvalues are bifurcation points and that the bifurcation solution is unique provided that two specified hypotheses are valid. The asymptotic formulae for the non-axisymmetric bifurcation solution are given and curves showing the corresponding deflection and stress are presented.

2. Mathematical description of the problem

We consider a polar orthotropic annular plate with thickness h, and inner and outer radii a and b. It is assumed that the inner and outer edges are subjected to inplane uniform radial pressures p_a and p_b, respectively, and that the anisotropic polar point of the plate coincides with the geometric centre O of the annulus. We take O as the origin of polar coordinates (r, θ) and the mid-plane as the (r, θ)-plane, and assume that, when the annular plate is in its unbuckled state, it occupies the region Ω in the (r, θ)-plane and its edge is $\partial\Omega$. We denote the radial and tangential displacements of the mid-plane of the plate by u_r and u_θ, the deflection by $\bar{w}$, the stress components by σ_r, σ_θ, $\tau_{r\theta}$, the strain components by ε_r, ε_θ, $\gamma_{r\theta}$ and the bending moments and shearing forces by M_r, M_θ, $M_{r\theta}$ and Q_r, Q_θ, respectively. These quantities are all functions of r and θ. In the region Ω, we have the following equations:

(i) Equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0,$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} = 0,$$

$$\frac{\partial \sigma_r}{\partial r}\frac{\partial \bar{w}}{\partial r} + \frac{\partial \sigma_\theta}{\partial \theta}\left(\frac{1}{r^2}\frac{\partial \bar{w}}{\partial \theta}\right) + \frac{\partial \tau_{r\theta}}{\partial r}\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial \theta}\right) + \frac{\partial \tau_{r\theta}}{\partial \theta}\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial r}\right)$$

$$= \sigma_r\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2}\right) + \sigma_\theta\left(\frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2}\right) + \tau_{r\theta}\left(\frac{2}{r}\frac{\partial^2 \bar{w}}{\partial r\,\partial \theta}\right).$$

(ii) Strain-displacement relations

$$\varepsilon_r = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial r}\right)^2,$$

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial \theta}\right)^2,$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r}\frac{\partial \bar{w}}{\partial \theta}.$$

(iii) Constitutive equations

$$\varepsilon_r = \frac{1}{E_r}(\sigma_r - \nu_r\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E_\theta}(\sigma_\theta - \nu_\theta\sigma_r), \quad \gamma_{r\theta} = \frac{1}{G}\tau_{r\theta},$$

in which E_r, E_θ, ν_r, ν_θ and G are material constants and satisfy the relation $\nu_\theta/\nu_r = E_\theta/E_r$. As in [6, 7], for the present case, there exists a single-valued stress function $\phi(r, \theta)$ in terms of which the stress may be expressed as

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta}.$$

From these relations, we obtain the following governing equations for buckling of polar orthotropic annular plates:

$$D_r \frac{\partial^4 \bar{w}}{\partial r^4} + D_r \frac{2}{r}\frac{\partial^3 \bar{w}}{\partial r^3} - D_\theta \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial r^2} + D_\theta \frac{1}{r^3}\frac{\partial \bar{w}}{\partial r} + D_\theta \frac{1}{r^4}\frac{\partial^4 \bar{w}}{\partial \theta^4} + 2(D_\theta + D_k)\frac{1}{r^4}\frac{\partial^2 \bar{w}}{\partial \theta^2}$$

$$+ 2D_k \frac{1}{r^2}\frac{\partial^4 \bar{w}}{\partial r^2 \partial \theta^2} - 2D_k \frac{1}{r^3}\frac{\partial^3 \bar{w}}{\partial r \partial \theta^2} - h\left[\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{\partial^2 \phi}{\partial r^2}\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2}\right)\right.$$

$$\left. + 2\left(\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 \bar{w}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \bar{w}}{\partial \theta}\right)\right] = 0, \tag{2.1$_a$}$$

$$\frac{1}{E_\theta}\frac{\partial^4 \phi}{\partial r^4} + \frac{1}{E_\theta}\frac{2}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{1}{E_r}\frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{E_r}\frac{1}{r^3}\frac{\partial \phi}{\partial r} + \frac{1}{E_r}\frac{1}{r^4}\frac{\partial^4 \phi}{\partial \theta^4} + \left(\frac{1}{2G} + \frac{1 - \nu_r}{E_r}\right)\frac{2}{r^4}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$+ \left(\frac{1}{G} - \frac{2\nu_r}{E_r}\right)\frac{1}{r^2}\frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} + \left(\frac{2\nu_r}{E_r} - \frac{1}{G}\right)\frac{1}{r^3}\frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{2}{r^3}\frac{\partial^2 \bar{w}}{\partial r \partial \theta}\frac{\partial \bar{w}}{\partial \theta} - \frac{1}{r^4}\left(\frac{\partial \bar{w}}{\partial \theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 \bar{w}}{\partial r \partial \theta}\right)^2$$

$$+ \frac{1}{r}\frac{\partial^2 \bar{w}}{\partial r^2}\frac{\partial \bar{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2} = 0, \tag{2.1$_b$}$$

in which

$$D_r = \frac{E_r h^3}{12(1 - \nu_r \nu_\theta)}, \quad D_\theta = \frac{E_\theta h^3}{12(1 - \nu_r \nu_\theta)}, \quad D_k = \nu_\theta D_r + D_{r\theta}, \quad D_{r\theta} = \frac{Gh^3}{6}.$$

The boundary conditions in the transverse direction at the edges of the annular plate may be any of the following conditions (or appropriate combinations of them):

$$\text{(a)} \quad \bar{w} = \frac{\partial \bar{w}}{\partial r} = 0, \qquad \text{for edges fixed;}$$

$$\text{(b)} \quad \bar{w} = M_r = 0, \qquad \text{for edges simply supported;} \tag{2.2}$$

$$\text{(c)} \quad M_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + N\frac{\partial \bar{w}}{\partial r} = 0, \quad \text{for edges free,}$$

where N is the assigned normal membrane force on an edge.

The boundary conditions in the mid-plane are

$$\begin{aligned} \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} &= -p_a, \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0, \quad \text{for } r = a, \\ \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} &= -p_b, \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0, \quad \text{for } r = b. \end{aligned} \tag{2.3}$$

For perforated plates, we have to add conditions which ensure the displacements to be single-valued. From [6], we have

THEOREM 1: *In the annular region Ω, there exist single-valued displacements u, v and rotation $\omega = \partial v/\partial x - \partial u/\partial y$ if and only if the following relations*

$$\begin{aligned}
&\int_0^{2\pi}\left(\frac{\partial\hat{\varepsilon}_\theta}{\partial r}r - \hat{\varepsilon}_r + \hat{\varepsilon}_\theta - \frac{1}{2}\frac{\partial\hat{\gamma}_{r\theta}}{\partial\theta}\right)\bigg|_{r=a} d\theta = 0,\\
&\int_0^{2\pi}\left(-\frac{\partial\hat{\gamma}_{r\theta}}{\partial\theta} + r\frac{\partial\hat{\varepsilon}_\theta}{\partial r} - \hat{\varepsilon}_r\right)\bigg|_{r=a} \cos\theta\, d\theta = 0,\\
&\int_0^{2\pi}\left(-\frac{\partial\hat{\gamma}_{r\theta}}{\partial\theta} + r\frac{\partial\hat{\varepsilon}_\theta}{\partial r} - \hat{\varepsilon}_r\right)\bigg|_{r=a} \sin\theta\, d\theta = 0
\end{aligned} \tag{2.4}$$

are valid, where

$$\hat{\varepsilon}_r \equiv \varepsilon_r - \frac{1}{2}\left(\frac{\partial\bar{w}}{\partial r}\right)^2, \quad \hat{\varepsilon}_\theta \equiv \varepsilon_\theta - \frac{1}{2}\left(\frac{1}{r}\frac{\partial\bar{w}}{\partial\theta}\right)^2, \quad \hat{\gamma}_{r\theta} \equiv \gamma_{r\theta} - \frac{1}{r}\frac{\partial\bar{w}}{\partial\theta}\frac{\partial\bar{w}}{\partial r},$$

$$u = u_r\cos\theta - u_\theta\sin\theta, \quad v = u_r\sin\theta + u_\theta\cos\theta.$$

By using the relations between the stress function ϕ and the stress, the relations (2.4) may be expressed in terms of ϕ and $\bar{w}$.

When the stress states are completely defined, the difference between any two stress functions is an arbitrary linear function of x and y. In order to uniquely define the stress function we have to specify some normalizing conditions, for example,

$$\phi(b, 0) = \phi(b, \pi) = \frac{\partial\phi}{\partial\theta}(b, 0) = 0. \tag{2.5}$$

Thus, the problem is to determine the functions $\bar{w}(r, \theta)$ and $\phi(r, \theta)$ in Ω such that equations (2.1) and conditions (2.2)–(2.5) are satisfied.

By introducing non-dimensional variables

$$\begin{aligned}
&x = \frac{r}{b}, \quad c = \frac{a}{b} < 1, \quad w(x, \theta) = \frac{C_2}{h}\bar{w}(r, \theta), \quad \tilde{F}(x, \theta) = \frac{C_2^2}{E_r h^2}\phi(r, \theta),\\
&C_2^2 = 12(1 - \nu_r\nu_\theta), \quad \alpha = \frac{\nu_\theta D_r + D_{r\theta}}{D_r}, \quad \beta = \frac{E_\theta}{E_r} = \frac{\nu_\theta}{\nu_r},\\
&\delta = \beta\nu_r - \frac{E_\theta}{2G}, \quad \bar{k} = \frac{p_b}{p_a}, \quad \lambda = C_2^2\frac{p_a}{E_r}\left(\frac{b}{h}\right)^2,
\end{aligned} \tag{2.6}$$

the boundary value problem (2.1)–(2.5) is reduced to the following non-dimensional form:

(i) Governing equations

$$L_1(w) - N_1(w, \tilde{F}) = 0,$$
$$L_2(\tilde{F}) + \frac{\beta}{2} N_1(w, w) = 0, \tag{2.7}$$

in which $L_i(\cdot)$ and $N_1(\cdot, :)$ are linear and non-linear differential operators, respectively, defined by

$$L_1(\cdot) \equiv \left[\frac{\partial^4}{\partial x^4} + \frac{2}{x}\frac{\partial^3}{\partial x^3} - \frac{\beta}{x^2}\frac{\partial^2}{\partial x^2} + \frac{\beta}{x^3}\frac{\partial}{\partial x} + \frac{\beta}{x^4}\frac{\partial^4}{\partial \theta^4} + \frac{2(\alpha+\beta)}{x^4}\frac{\partial^2}{\partial \theta^2} + \frac{2\alpha}{x^2}\frac{\partial^4}{\partial x^2 \partial \theta^2} - \frac{2\alpha}{x^3}\frac{\partial^3}{\partial x \partial \theta^2}\right](\cdot),$$

$$L_2(\cdot) \equiv \left[\frac{\partial^4}{\partial x^4} + \frac{2}{x}\frac{\partial^3}{\partial x^3} - \frac{\beta}{x^2}\frac{\partial^2}{\partial x^2} + \frac{\beta}{x^3}\frac{\partial}{\partial x} + \frac{\beta}{x^4}\frac{\partial^4}{\partial \theta^4} + \frac{2(\beta-\delta)}{x^4}\frac{\partial^2}{\partial \theta^2} - \frac{2\delta}{x^2}\frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{2\delta}{x^3}\frac{\partial^3}{\partial x \partial \theta^2}\right](\cdot), \tag{2.8}$$

$$N_1(\cdot, :) \equiv \left(\frac{1}{x}\frac{\partial(\cdot)}{\partial x} + \frac{1}{x^2}\frac{\partial^2(\cdot)}{\partial \theta^2}\right)\frac{\partial^2(:)}{\partial x^2} + \left(\frac{1}{x}\frac{\partial(:)}{\partial x} + \frac{1}{x^2}\frac{\partial^2(:)}{\partial \theta^2}\right)\frac{\partial^2(\cdot)}{\partial x^2} - 2\left(\frac{1}{x^2}\frac{\partial(\cdot)}{\partial \theta} - \frac{1}{x}\frac{\partial^2(\cdot)}{\partial x \partial \theta}\right)\left(\frac{1}{x^2}\frac{\partial(:)}{\partial \theta} - \frac{1}{x}\frac{\partial^2(:)}{\partial x \partial \theta}\right);$$

(ii) Boundary conditions in the transverse direction

$$\text{(a)} \quad w = \frac{\partial w}{\partial x} = 0, \qquad \text{for } x = c, 1,$$

$$\text{(b)} \quad w = \tilde{M}_x = 0, \qquad \text{for } x = c, 1, \tag{2.9)'$$

$$\text{(c)} \quad \tilde{M}_x = \tilde{V}_x + \frac{Nb^2}{D_r}\frac{\partial w}{\partial x} = 0, \quad \text{for } x = c, 1,$$

or any appropriate combination of these boundary conditions, in which

$$\tilde{M}_x \equiv \frac{C_2 b^2}{D_r h} M_r = -\left[\frac{\partial^2 w}{\partial x^2} + \nu_0\left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right],$$

$$\tilde{M}_\theta \equiv \frac{C_2 b^2}{D_\theta h} M_\theta = -\left[\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2} + \nu_r \frac{\partial^2 w}{\partial x^2}\right],$$

$$\tilde{M}_{x0} \equiv \frac{C_2 b^2}{D_{r0} h} M_{r0} = -\frac{\partial^2}{\partial x \partial \theta}\left(\frac{w}{x}\right),$$

$$\tilde{V}_x \equiv \frac{C_2 b^2}{D_r h}\left(Q_r + \frac{1}{r}\frac{\partial M_{r0}}{\partial \theta}\right) \tag{2.10}$$

$$= -\left[\frac{\partial^3 w}{\partial x^3} + \frac{1}{x}\frac{\partial^2 w}{\partial x^2} + \frac{2\alpha - \nu_0}{x^2}\left(\frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{1}{x}\frac{\partial^2 w}{\partial \theta^2}\right) - \frac{\beta}{x^2}\left(\frac{\partial w}{\partial x} + \frac{1}{x}\frac{\partial^2 w}{\partial \theta^2}\right)\right].$$

For convenience, we write (2.9)′ in the unified form

$$G(w) = 0. \tag{2.9}$$

(iii) Boundary conditions in the mid-plane

$$\frac{1}{x}\frac{\partial \tilde{F}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2} = -\lambda, \quad \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \tilde{F}}{\partial \theta}\right) = 0, \quad \text{for } x = c,$$

$$\frac{1}{x}\frac{\partial \tilde{F}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2} = -\bar{k}\lambda, \quad \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \tilde{F}}{\partial \theta}\right) = 0, \quad \text{for } x = 1. \tag{2.11}$$

(iv) Single-valuedness conditions for the displacement

$$\int_0^{2\pi} [\Sigma_1(\tilde{F}) + \Lambda_1(w)]|_{x=c}\, d\theta = 0,$$

$$\int_0^{2\pi} [\Sigma_2(\tilde{F}) + \Lambda_2(w)]|_{x=c} \cos\theta\, d\theta = 0, \tag{2.12}$$

$$\int_0^{2\pi} [\Sigma_2(\tilde{F}) + \Lambda_2(w)]|_{x=c} \sin\theta\, d\theta = 0,$$

in which $\Sigma_i(\tilde{F})$ and $\Lambda_i(w)$ $(i = 1, 2)$ are linear and non-linear differential operators, defined by

$$\Sigma_1(\tilde{F}) \equiv \frac{\alpha(\nu_0 - \delta)}{\beta}\left(\frac{1}{x}\frac{\partial^3 \tilde{F}}{\partial x \partial \theta^2} - \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2}\right) - 2\left(\frac{1}{x}\frac{\partial \tilde{F}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2}\right)$$

$$+ \frac{2}{\beta}\left(x\frac{\partial^3 \tilde{F}}{\partial x^3} - \nu_0 \frac{1}{x}\frac{\partial^3 \tilde{F}}{\partial x \partial \theta^2} + \frac{\partial^2 \tilde{F}}{\partial x^2}\right),$$

$$\Sigma_2(\tilde{F}) \equiv (\nu_0 - \delta)\left(\frac{1}{x}\frac{\partial^3 \tilde{F}}{\partial x \partial \theta^2} - \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2}\right) - \frac{\beta}{2}\left(\frac{1}{x}\frac{\partial \tilde{F}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2}\right)$$

$$+ \frac{1}{2}\left[x\frac{\partial^3 \tilde{F}}{\partial x^3} + \nu_0\left(\frac{1}{x}\frac{\partial \tilde{F}}{\partial x} + \frac{2}{x^2}\frac{\partial^2 \tilde{F}}{\partial \theta^2}\right) - \frac{1}{x}\frac{\partial^3 \tilde{F}}{\partial x \partial \theta^2}\right], \tag{B}$$

$$\Lambda_1(w) \equiv \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right)^2 + \frac{1}{x}\frac{\partial^2 w}{\partial \theta^2}\frac{\partial w}{\partial x} - \frac{1}{x}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial x \partial \theta},$$

$$\Lambda_2(w) \equiv \frac{\beta}{2}\left[\frac{1}{x}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial \theta^2} + \left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right].$$

As $\Lambda_i(w)$ ($i = 1, 2$) are non-linear in w, the single-valuedness conditions (2.12) are three non-linear functional constraint equations.

(v) Normalizing conditions

$$\tilde{F}(1, 0) = \tilde{F}(1, \pi) = \frac{\partial \tilde{F}}{\partial \theta}(1, 0) = 0. \tag{2.13}$$

Thus, the problem is reduced to solving for the functions $w(x, \theta)$ and $\tilde{F}(x, \theta)$ in the region: $c \leqslant x \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, such that the non-linear equations (2.7) and the conditions (2.9)–(2.13) are all satisfied.

Let the solution for unbuckled states of the annular plate be (w^0, F^0). By definition, letting $w = w^0 = 0$ in (2.7)–(2.13), we have a boundary-value problem for F^0. According to the uniqueness theorem of elasticity theory, the solution $F^0(x, \theta) \equiv F^0(x)$ is unique.

When $\beta = 1$, we obtain from (2.7)–(2.13)

$$F^0(x; \lambda) = \tilde{C}_1 + \tilde{C}_2 \ln x + \tilde{C}_3 x^2 + \tilde{C}_4 x^2 \ln x, \tag{2.14$_a$}$$

in which

$$\tilde{C}_1 = -C_3, \quad \tilde{C}_2 = \frac{(k-1)c^2}{1-c^2}\lambda, \quad \tilde{C}_3 = \frac{c^2 - \bar{k}}{2(1-c^2)}\lambda, \quad \tilde{C}_4 = 0;$$

when $\beta \neq 1$, we have

$$F^0(x; \lambda) = \tilde{C}_1 + \tilde{C}_2 x^2 + \tilde{C}_3 x^{1+\sqrt{\beta}} + \tilde{C}_4 x^{1-\sqrt{\beta}}, \tag{2.14$_b$}$$

in which

$$\tilde{C}_1 = -(\tilde{C}_2 + \tilde{C}_3 + \tilde{C}_4), \quad \tilde{C}_2 = 0, \quad \tilde{C}_3 = \frac{(c^{-\sqrt{\beta}}\bar{k} - c)\lambda}{(1+\sqrt{\beta})(c^{\sqrt{\beta}} - c^{-\sqrt{\beta}})},$$

$$\tilde{C}_4 = \frac{(c - \bar{k}c^{-\sqrt{\beta}})\lambda}{(1-\sqrt{\beta})(c^{\sqrt{\beta}} - c^{-\sqrt{\beta}})},$$

and we note that the expression for $F^0(x)$ has to be suitably modified when $p_a = 0$.

Now let $F(x, \theta) \equiv \tilde{F}(x, \theta) - F^0(x, \theta)$, then the boundary-value problem (2.7)–(2.13) may be rewritten as follows:

(i) Governing equations

$$L_1(w) - \frac{1}{x}\frac{\mathrm{d}F^0}{\mathrm{d}x}\frac{\partial^2 w}{\partial x^2} - \left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2}\right)\frac{\mathrm{d}^2 F^0}{\mathrm{d}x^2} - N_1(w, F) = 0,$$

$$L_2(F) + \frac{\beta}{2} N_1(w, w) = 0; \tag{2.15}$$

(ii) Boundary conditions in the transverse direction

$$G(w) = 0; \tag{2.16}$$

(iii) Boundary conditions in the mid-plane

$$\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial \theta}\right) = 0, \quad \text{for } x = c, 1; \tag{2.17}$$

(iv) Single-valuedness conditions of the displacement

$$\int_0^{2\pi} [\Sigma_1(F) + \Lambda_1(w)]|_{x=c}\, \mathrm{d}\theta = 0,$$

$$\int_0^{2\pi} [\Sigma_2(F) + \Lambda_2(w)]|_{x=c} \cos\theta\, \mathrm{d}\theta = 0, \tag{2.18}$$

$$\int_0^{2\pi} [\Sigma_2(F) + \Lambda_2(w)]|_{x=c} \sin\theta\, \mathrm{d}\theta = 0;$$

(v) Normalizing conditions

$$F(1, 0) = F(1, \pi) = \frac{\partial F}{\partial \theta}(1, 0) = 0. \tag{2.19}$$

Therefore, the fundamental problem (which we denote as (EP)) is finally reduced to determining $w(x, \theta)$ and $F(x, \theta)$ such that the equations (2.15) and the conditions (2.16)–(2.19) are satisfied.

3. Linearized problem and critical loads

Obviously, for any λ, the problem (EP) has the trivial solution

$$w(x, \theta; \lambda) = 0, \quad F(x, \theta; \lambda) = 0,$$

which corresponds to unbuckled states of the annular plate. As is known from bifurcation theory [8], non-trivial solutions branch from the trivial solution of the problem (EP) only at eigenvalues λ^* for which the linearized problem (which we denote by (LP)) has at least one non-zero solution; that is, the following boundary-value problem has at least one non-zero solution:

(i) Differential equations

$$L_1(\hat{w}) - \frac{1}{x}\frac{\mathrm{d}F^0}{\mathrm{d}x}\frac{\partial^2 \hat{w}}{\partial x^2} - \left(\frac{1}{x}\frac{\partial \hat{w}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \hat{w}}{\partial \theta^2}\right)\frac{\mathrm{d}^2 F^0}{\mathrm{d}x^2} = 0,$$

$$L_2(\hat{F}) = 0; \tag{3.1}$$

(ii) Boundary conditions in the transverse direction

$$G(\hat{w}) = 0; \tag{3.2}$$

(iii) Boundary conditions for the mid-plane

$$\frac{1}{x}\frac{\partial \hat{F}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \hat{F}}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \hat{F}}{\partial \theta}\right) = 0, \quad \text{for } x = c,\ 1; \tag{3.3}$$

(iv) Single-valuedness conditions for the displacement

$$\int_0^{2\pi} \Sigma_1(\hat{F})|_{x=c}\,\mathrm{d}\theta = \int_0^{2\pi} \Sigma_2(\hat{F})|_{x=c}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi} \Sigma_2(\hat{F})|_{x=c}\sin\theta\,\mathrm{d}\theta = 0; \tag{3.4}$$

(v) Normalizing conditions

$$\hat{F}(1, 0) = \hat{F}(1, \pi) = \frac{\partial \hat{F}}{\partial \theta}(1, 0) = 0. \tag{3.5}$$

From $(3.1)_b$ and (3.3)–(3.5), we obtain that $\hat{F}(x, \theta) = 0$. As the set $\{\cos n\theta\}$ is complete in the space $L_2(\Omega)$, any solution of equation $(3.1)_a$ with (3.2) may be written in separated variables form. Hence, we express $\hat{w}(x, \theta)$ as

$$\hat{w}(x, \theta) = \Sigma \hat{w}_n(x)\cos n\theta, \tag{3.6}$$

where n is the number of circumferential waves of the buckled annular plate for which the corresponding critical load takes the least value. When $n = 0$, buckled states of the annular plate are axisymmetric and when $n \neq 0$, buckled states are non-axisymmetric. The numerical computation shows that n is not always zero [1–5]. This means that an annular plate with axisymmetric edge conditions and subjected to axisymmetric inplane forces may give rise to non-axisymmetric buckling. Substituting (3.6) into $(3.1)_a$ and (3.2), we have the following boundary-value problem for $\hat{w}_n(x)$:

$$\tilde{L}_n(\hat{w}_n) \equiv x\left[\frac{\mathrm{d}^4}{\mathrm{d}x^4} + \frac{2}{x}\frac{\mathrm{d}^3}{\mathrm{d}x^3} - \frac{\beta + 2\alpha n^2}{x^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{\beta n^4 - 2(\alpha + \beta)n^2}{x^4}\right]\hat{w}_n$$

$$= \lambda\left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n^2}{x}\right)\right]\hat{w}_n \equiv Q, \tag{3.7}$$

$$\tilde{G}(\hat{w}_n) = 0, \tag{3.8}$$

in which $F^0(x, \lambda) \equiv \lambda \bar{F}^0(x)$, and $\tilde{G}(\cdot)$ is the result of substituting (3.6) into (2.10) and cancelling $\cos n\theta$.

Next, we use the shooting method [9, 10] to determine non-zero solutions of (3.7) and (3.8). For definiteness, we assume that the edge $x = 1$ is fixed. The treatment is similar when the edge $x = c$ is fixed.

First, we construct an initial-value problem

$$\tilde{L}_n(\psi) = Q, \quad \psi(1) = \psi'(1) = 0, \quad \psi''(1) = \alpha_1, \quad \psi'''(1) = \alpha_2, \tag{3.9}$$

in which α_1 and α_2 are undetermined constants. Obviously, any solution of (3.9) is a linear combination of solutions of the following two initial-value problems

$$\tilde{L}_n(\psi_1) = Q, \quad \psi_1(1) = \psi_1'(1) = 0, \quad \psi_1''(1) = 1, \quad \psi_1'''(1) = 0, \tag{3.10}$$

$$\tilde{L}_n(\psi_2) = Q, \quad \psi_2(1) = \psi_2'(1) = 0, \quad \psi_2''(1) = 0, \quad \psi_2'''(1) = 1. \tag{3.11}$$

For any given λ, the problems (3.10) and (3.11) have unique solutions $\psi_1(x; \lambda)$ and $\psi_2(x; \lambda)$, respectively. Hence, an arbitrary solution of (3.9) may be written as

$$\psi(x; \lambda) = \alpha_1\psi_1(x; \lambda) + \alpha_2\psi_2(x; \lambda). \tag{3.12}$$

To make $\psi(x; \lambda)$ a solution of (3.7) and (3.8), we must choose α_i such that the conditions at the edge $x = c$ are satisfied. Thus, we obtain a system of linear algebraic equations on α_1 and α_2, given by

$$[A]\{\alpha\} \equiv \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = 0. \tag{3.13}$$

The elements of the matrix A will be defined by the conditions at the edge $x = c$. For example, when the edge $x = c$ is clamped, we have

$$a_i = \psi_i(c; \lambda), \quad b_i = \psi_i'(c; \lambda), \quad i = 1, 2;$$

when the edge $x = c$ is simply supported, we have

$$a_i = \psi_i(c; \lambda), \quad b_i = \psi_i''(c; \lambda) + \frac{v_0}{c}\psi_i'(c; \lambda), \quad i = 1, 2;$$

and when the edge $x = c$ is free and $p_a = 0$, we have

$$a_i = \psi_i''(c; \lambda) + v_0\left(\frac{1}{c}\psi_i'(c; \lambda) - \frac{n^2}{c^2}\psi_i(c; \lambda)\right), \quad i = 1, 2;$$

$$b_i = \psi_i'''(c; \lambda) + \frac{1}{c}\psi_i''(c; \lambda) - \frac{(2\alpha - v_0)n^2 + \beta}{c^2}\psi_i'(c; \lambda) + \frac{2\alpha - v_0 + \beta}{c^3}n^2\psi_i(c; \lambda).$$

Table 1. Critical loads (for inner edge free and outer edge fixed, and $p_a = 0$ and $\nu_\theta = 0.3$)

c \ β	0.5	0.8	1	1.25	2	5
0.1	8.24 8.25* (0)	11.74 11.77* (0)	13.95 13.95* (0)	16.56 16.58* (0)	23.79 23.61* (0)	49.26 49.29* (0)
0.2	8.86 8.87* (0)	11.70 11.70* (0)	13.60 13.60* (0)	15.98 15.99* (0)	23.03 23.06* (0)	49.02 49.05* (0)
0.3	11.08 11.08* (0)	13.39 13.39* (0)	14.96 14.96* (0)	16.96 16.96* (0)	23.12 23.13* (0)	48.27 48.30* (0)
0.4	15.25 15.26* (0)	17.22 17.22* (0)	18.55 18.55* (0)	20.24 20.25* (0)	25.50 25.50* (0)	48.04 48.09* (0)
0.5	22.86 22.86* (0)	24.59 24.59* (0)	25.76 25.76* (0)	27.23 27.23* (0)	31.76 31.77* (0)	51.26 51.31* (0)
0.6	28.34 28.35* (3)	31.56 31.56* (3)	33.50 33.50* (2)	35.36 33.36* (2)	41.02 41.02* (2)	60.74 60.74* (1)
0.7	36.89 36.87* (5)	41.36 41.36* (5)	43.68 43.69* (4)	46.45 46.46* (4)	53.94 53.94* (3)	77.34 77.34* (2)
0.8	54.00 54.03* (9)	60.71 60.72* (8)	64.66 64.70* (7)	68.70 68.71* (7)	79.38 79.40* (6)	111.57 111.60* (5)
0.9	106.08 106.10* (21)	119.80 119.80* (18)	127.48 127.50* (17)	136.01 136.00* (16)	157.19 157.20* (14)	216.80 216.80* (11)

Remark: Quantities with * are results in [1] and the number in (·) is n.

Thus, (3.13) has non-zero solutions $\{\alpha\} \neq 0$ if and only if

$$a_1 b_2 - a_2 b_1 = 0. \tag{3.14}$$

Hence, λ is a critical load for the annular plate if and only if λ is a root of the characteristic equation (3.14). We use a finite-difference method with variable step-length to solve (3.7) and (3.8), and then apply (3.14) to give the critical loads on the annular plate for a variety of edge conditions. In Tables 1, 2 we list some numerical results for the critical loads and compare them with those of [1]. For later convenience, we also list some values for a_i and b_i in Table 3.

According to the general theory of ordinary differential equations, when $\lambda = \lambda^*$ is a root of (3.14), the linearized problem (LP) has at most two non-zero solutions which are linearly independent. But according to the theory of linear algebra, when $\lambda = \lambda^*$ is a root of (3.14), there exist two linearly independent solutions to (LP) if and only if λ^* is simultaneously an eigenvalue of the following four problems:

(i) $\tilde{L}_n(z) = Q, \quad z(1) = z'(1) = z'''(1) = a_1 = 0,$

Table 2. Critical loads (for edges fixed, and $p_a = 0$ and $v_\theta = 0.3$)

c \ β	0.5	0.8	1	1.25	2	5
0.1	38.70 38.71* (2)	42.25 42.25* (2)	44.52 44.53* (2)	47.40 47.42* (2)	55.73 55.74* (2)	84.34 84.30* (1)
0.2	50.19 50.19* (3)	53.81 53.83* (2)	55.68 55.69* (2)	58.01 58.01* (2)	64.87 64.89* (2)	91.67 91.68* (2)
0.3	62.38 62.39* (4)	67.71 67.71* (3)	70.24 70.25* (3)	73.40 73.41* (3)	82.85 82.86* (3)	107.76 107.76* (3)
0.4	77.47 77.48* (5)	85.52 85.53* (5)	89.11 89.11* (4)	93.16 93.17* (4)	105.1 105.1* (3)	134.2 134.2* (3)
0.5	97.39 97.39* (7)	108.1 108.2* (6)	113.8 113.8* (6)	120.71 120.70* (6)	134.70 134.70* (5)	173.29 173.30* (4)
0.6	126.18 126.20* (10)	141.58 141.60* (9)	150.01 150.00* (6)	158.79 158.80* (8)	180.18 180.20* (7)	233.39 233.40* (5)
0.7	173.20 173.30* (15)	196.09 196.10* (13)	208.67 208.70* (12)	222.29 222.30* (12)	254.70 254.70* (10)	337.58 337.60* (8)
0.8	266.59 266.60* (24)	303.88 303.90* (22)	324.40 324.40* (20)	346.89 346.90* (19)	402.01 402.00* (17)	544.69 544.70* (13)
0.9	544.60 544.60* (53)	624.69 624.70* (44)	669.19 669.20* (44)	716.69 716.70* (42)	840.78 840.80* (37)	1168.30 1168.00* (29)

Remark: Quantities with * are results in [1] and the number in $(\cdot)$ is n.

Table 3. Values of a_i (for inner edge free and outer edge fixed, and $p_a = 0$ and $v_\theta = 0.3$)

c	β	0.5	1	2	c	β	0.5	1	5
0.6 (2)	a_1	1.325	1.315	1.298	0.8 (7)	a_1	4.973	4.917	4.512
	a_2	−0.733	−0.736	−0.742		a_2	−0.489	−0.487	−0.469
0.7 (4)	a_1	3.711	3.657	3.554	0.9 (9)	a_1	2.203	2.198	2.165
	a_2	−0.707	−0.704	−0.698		a_2	−0.145	−0.145	−0.144

(ii) $\tilde{L}_n(z) = Q, \quad z(1) = z'(1) = z'''(1) = b_1 = 0,$

(iii) $\tilde{L}_n(z) = Q, \quad z(1) = z'(1) = z''(1) = a_2 = 0,$

(iv) $\tilde{L}_n(z) = Q, \quad z(1) = z'(1) = z''(1) = b_2 = 0.$

From Table 3 it is easily seen that a_i and b_i are not all zero. This means that generally there is no λ^* which is simultaneously an eigenvalue of the above four problems.

HYPOTHESIS 1: There is no λ which is simultaneously an eigenvalue of the above four problems.

THEOREM 2: *If* $\lambda = \lambda^*$ *is one eigenvalue of* (3.7) *and* (3.8), *and Hypothesis* 1 *is valid, the only non-zero solution to the problem* (3.7) *and* (3.8) *is* (3.12).

In this case, for some N which corresponds to $\lambda = \lambda^*$, the eigenvector of the linearized problem (LP) is $(\hat{w}_N, \hat{F}) = (\hat{w}_N(x) \cos N\theta, 0)$, and the dimension of the space of eigenvectors is 1; that is, the space of eigenvectors of (LP) is spanned by $\{\hat{w}_N(x) \cos N\theta, 0\}$.

4. Post-buckling analysis and asymptotic formulae of bifurcation solutions

To discuss non-trivial solutions of the problem (EP) near $\lambda = \lambda^*$, we seek a solution of the following form:

$$W_N(x, \theta) = \varepsilon \hat{w}_N(x, \theta) + \varepsilon \bar{W}_N(x, \theta; \varepsilon),$$

$$F_N(x, \theta) = \varepsilon \bar{F}_N(x, \theta; \varepsilon), \quad \lambda = \lambda^* + \bar{\lambda}_N(\varepsilon). \tag{4.1}$$

Here, we have set $\hat{w}_N(x, \theta) \equiv \hat{w}_N(x) \cos N\theta$, where N is the number of circumferential waves of the buckled plate when $\lambda = \lambda^*$ and $\hat{w}_N(x, \theta)$ is the non-zero solution of (3.7) and (3.8); ε in (4.1) is a small parameter defined by

$$\varepsilon = \int_0^{2\pi} \int_c^1 W_N \hat{w}_N \, \mathrm{d}x \, \mathrm{d}\theta \Big/ \int_0^{2\pi} \int_c^1 \hat{w}_N^2 \, \mathrm{d}x \, \mathrm{d}\theta. \tag{4.2}$$

Hence, we immediately obtain

$$\int_0^{2\pi} \int_c^1 \bar{W}_N(x, \theta) \hat{w}_N(x, \theta) \, \mathrm{d}x \, \mathrm{d}\theta = 0. \tag{4.3}$$

By substituting (4.1) into the equations and conditions (2.15)–(2.19) for the problem (EP), we obtain the equations and boundary conditions for $\bar{W}_N$, $\bar{F}_N$ and $\bar{\lambda}_N$ as follows:

(i) Differential equations

$$L_{\lambda^*}(\bar{W}_N) \equiv L_1(\bar{W}_N) - \lambda^* \left[\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\partial^2 \bar{W}_N}{\partial x^2} + \left(\frac{1}{x}\frac{\partial \bar{W}_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \bar{W}_N}{\partial \theta^2}\right)\frac{\mathrm{d}^2 \bar{F}^0}{\mathrm{d}x^2}\right]$$

$$= \bar{\lambda}_N \left[\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\partial^2(\hat{w}_N + \bar{W}_N)}{\partial x^2} + \left(\frac{1}{x^2}\frac{\partial^2(\hat{w}_N + \bar{W}_N)}{\partial \theta^2} + \frac{1}{x}\frac{\partial(\hat{w}_N + \bar{W}_N)}{\partial x}\right)\frac{\mathrm{d}^2 \bar{F}^0}{\mathrm{d}x^2}\right]$$

$$+ \varepsilon N_1(\bar{F}_N, \hat{w}_N + \bar{W}_N), \tag{4.4$_a$}$$

$$L_2(\bar{F}_N) = -\frac{\varepsilon}{2}\beta N_1(\hat{w}_N + \bar{W}_N, \hat{w}_N + \bar{W}_N); \tag{4.4$_b$}$$

(ii) Boundary conditions in the transverse direction

$$G(\bar{W}_N) = 0; \tag{4.5}$$

(iii) Boundary conditions in the mid-plane

$$\frac{1}{x}\frac{\partial \bar{F}_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \bar{F}_N}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \bar{F}_N}{\partial \theta}\right) = 0, \quad \text{for } x = c,\ 1; \tag{4.6}$$

(iv) Single-valuedness conditions of the displacement

$$\begin{aligned} &\int_0^{2\pi} [\Sigma_1(\bar{F}_N) + \varepsilon\Lambda_1(\hat{w}_N + \bar{W}_N)]|_{x=c}\,\mathrm{d}\theta = 0, \\ &\int_0^{2\pi} [\Sigma_2(\bar{F}_N) + \varepsilon\Lambda_2(\hat{w}_N + \bar{W}_N)]|_{x=c}\cos\theta\,\mathrm{d}\theta = 0, \\ &\int_0^{2\pi} [\Sigma_2(\bar{F}_N) + \varepsilon\Lambda_2(\hat{w}_N + \bar{W}_N)]|_{x=c}\sin\theta\,\mathrm{d}\theta = 0; \end{aligned} \tag{4.7}$$

(v) Normalizing conditions

$$\bar{F}_N(1, 0) = \bar{F}_N(1, \pi) = \frac{\partial \bar{F}_N}{\partial \theta}(1, 0) = 0. \tag{4.8}$$

If for any ε: $0 < |\varepsilon| < \varepsilon_0$, we can obtain one unique solution $\bar{W}_N$, $\bar{F}_N$ and $\bar{\lambda}_N$ to the problem (4.3)–(4.8) satisfying

$$\lim_{\varepsilon\to 0} \bar{W}_N(x, \theta; \varepsilon) = \lim_{\varepsilon\to 0} \bar{F}_N(x, \theta; \varepsilon) = \lim_{\varepsilon\to 0} \bar{\lambda}_N(\varepsilon) = 0, \tag{4.9}$$

the solution (4.1) is just one non-trivial solution of (EP) near $\lambda = \lambda^*$ and it branches from the trivial solution of (EP). Hence, the solution (4.1) is just one bifurcation solution near $\lambda = \lambda^*$ and describes the buckling and post-buckling behaviour of the annular plate. It is easy to see that the problem (4.3)–(4.8) has the trivial solution $\bar{W}_N = \bar{F}_N = \bar{\lambda}_N = 0$ when $\varepsilon = 0$. Therefore, to prove that the problem (4.3)–(4.8) has a unique solution $\bar{W}_N$, $\bar{F}_N$ and $\bar{\lambda}_N$ satisfying the condition (4.9), it is sufficient to prove only that $\varepsilon = 0$ is not an eigenvalue of the linearized problem. In other words, we have to show that the only solution to the following problem

$$L_{\lambda^*}(\tilde{W}_N) - \lambda_N\left[\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\partial^2 \hat{w}_N}{\partial x^2} + \left(\frac{1}{x}\frac{\partial \hat{w}_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \hat{w}_N}{\partial \theta^2}\right)\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\right] = 0,$$

$$L_2(\tilde{F}_N) = 0,$$

$$G(\tilde{W}_N) = 0,$$

$$\frac{1}{x}\frac{\partial \tilde{F}_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \tilde{F}_N}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \tilde{F}_N}{\partial \theta}\right) = 0, \quad \text{for } x = c,\ 1$$

$$\int_0^{2\pi} \Sigma_1(\tilde{F}_N)|_{x=c}\,\mathrm{d}\theta = \int_0^{2\pi} \Sigma_2(\tilde{F}_N)|_{x=c} \cos\theta\,\mathrm{d}\theta = \int_0^{2\pi} \Sigma_2(\tilde{F}_N)|_{x=c} \sin\theta\,\mathrm{d}\theta = 0,$$

$$\tilde{F}_N(1, 0) = \tilde{F}_N(1, \pi) = \frac{\partial \tilde{F}_N}{\partial\theta}(1, 0) = 0.$$

is the zero-solution $\tilde{W}_N = \tilde{F}_N = \tilde{\lambda}_N = 0$. It is not difficult to show that $\tilde{F}_N(x, \theta) = 0$. To solve for $\tilde{W}_N(x, \theta)$, we also assume that

$$\tilde{W}_N(x, \theta) \equiv \tilde{W}_N(x) \cos N\theta.$$

By substituting this into the equation and boundary conditions for $\tilde{W}_N(x, \theta)$, we obtain the boundary-value problem for $\tilde{W}_N(x)$

$$\tilde{L}_{\lambda^*}(\tilde{W}_N(x)) \equiv \tilde{L}_N(\tilde{W}_N) - \lambda^* \left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\mathrm{d}^2\tilde{W}_N}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}\tilde{W}_N}{\mathrm{d}x} - \frac{N^2}{x}\tilde{W}_N\right)\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\right]$$

$$= \tilde{\lambda}_N \left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\mathrm{d}^2\hat{w}_N}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}\hat{w}_N}{\mathrm{d}x} - \frac{N^2}{x}\hat{w}_N\right)\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\right], \tag{4.9}$$

$$\tilde{G}(\tilde{W}_N) = 0.$$

By multiplying both sides of $(4.9)_a$ by $\hat{w}_N(x)$ and integrating from c to 1, we obtain

$$\int_c^1 \tilde{L}_{\lambda^*}(\hat{w}_N)\tilde{W}_N(x)\,\mathrm{d}x - \tilde{\lambda}_N \int_c^1 \left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\left(\frac{\mathrm{d}\hat{w}_N}{\mathrm{d}x}\right)^2 + \frac{N^2}{x}\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\hat{w}_N^2\right]\mathrm{d}x$$

$$-\lambda^* \left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\left(\hat{w}_N\frac{\mathrm{d}\tilde{W}_N}{\mathrm{d}x} - \frac{\mathrm{d}\hat{w}_N}{\mathrm{d}x}\tilde{W}_N\right)\right]_{x=c}^{x=1} - \tilde{\lambda}_N\left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\hat{w}_N\frac{\mathrm{d}\hat{w}_N}{\mathrm{d}x}\right]_{x=c}^{x=1} = 0. \tag{4.10}$$

As $\hat{w}_N(x)$ satisfies the equation (3.7), $\tilde{L}_{\lambda^*}(\hat{w}_N(x)) = 0$. At the same time, if the given edge conditions require that

$$\left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\hat{w}_N\frac{\mathrm{d}\hat{w}_N}{\mathrm{d}x}\right]_{x=c}^{x=1} = 0,$$

and hence that,

$$\left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\hat{w}_N\frac{\mathrm{d}\tilde{W}_N}{\mathrm{d}x}\right]_{x=c}^{x=1} = \left[\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\mathrm{d}\hat{W}_N}{\mathrm{d}x}\tilde{w}_N\right]_{x=c}^{x=1} = 0,$$

then from (4.10), we derive that $\tilde{\lambda}_N = 0$ provided that the sign of $\mathrm{d}\bar{F}^0/\mathrm{d}x$ is the same as that of $\mathrm{d}^2\bar{F}^0/\mathrm{d}x^2$. The numerical computation shows that buckling does not take place when the unbuckled states of the annular plate are in tension. Therefore, we require that $\mathrm{d}\bar{F}^0/\mathrm{d}x \leqslant 0$ and $\mathrm{d}^2\bar{F}^0/\mathrm{d}x^2 \leqslant 0$. Comparing (4.9) with (3.7) and (3.8), and noticing Theorem 2, we find that $\tilde{W}_N(x, \theta) = \alpha_0 \hat{w}_N(x, \theta)$ and α_0 is a constant. But from (4.3), we derive that $\alpha_0 = 0$. Hence, $\tilde{W}_N(x, \theta) \equiv 0$. Thus, we have shown that the following theorem [8] holds:

THEOREM 3: *Assume that λ^* is an eigenvalue of* (3.7) *and* (3.8), *and Hypothesis* 1 *is valid. If unbuckled states of the annular plate are compressive, under appropriate boundary conditions, the only bifurcation solutions branch from the trivial solution of* (EP) *near $\lambda = \lambda^*$, that is, all eigenvalues λ^* are bifurcation points and the bifurcation solutions have the form* (4.1).

This theorem shows that non-axisymmetric buckled states of the annular plate with axisymmetric edge conditions can branch from axisymmetric unbuckled states under certain conditions. Next, we shall seek the bifurcation solution $\bar{W}_N$, $\bar{F}_N$ and $\bar{\lambda}_N$ of the problem (EP). Assume that the inner edge of the annular plate is free and the outer edge is fixed and subjected to a uniform radial pressure p_b. In this case, we have

$$F^0(x) \equiv \lambda \bar{F}^0(x)$$

$$= \begin{cases} \dfrac{\lambda}{2(1 - c^2)}(1 - x^2 + 2c^2 \ln x), & \text{for } \beta = 1, \\ \dfrac{\lambda}{c^{\sqrt{\beta}} - c^{-\sqrt{\beta}}}\left[\dfrac{c^{-\sqrt{\beta}}}{1 + \sqrt{\beta}}(x^{1+\sqrt{\beta}} - 1) - \dfrac{c^{\sqrt{\beta}}}{1 - \sqrt{\beta}}(x^{1-\sqrt{\beta}} - 1)\right], & \text{for } \beta \neq 1. \end{cases}$$

It is easy to verify that the conditions of Theorem 3 are satisfied. Assume that

$$\bar{W}_N = \Sigma \varepsilon^m \bar{W}_{Nm}(x, \theta), \quad \bar{F}_N = \Sigma \varepsilon^m \bar{F}_{Nm}(x, \theta), \quad \bar{\lambda}_N = \Sigma \varepsilon^m \bar{\lambda}_{Nm}.$$

By substituting these into (4.3)–(4.8) and comparing the coefficients of like powers in ε, we obtain boundary-value problems for $\bar{W}_{Nm}$, $\bar{F}_{Nm}$ and $\bar{\lambda}_{Nm}$ $(m = 1, 2, \ldots)$. On solving these problems, we get

$$\bar{W}_{N1}(x, \theta) = 0, \quad \bar{F}_{N2}(x, \theta) = 0, \quad \bar{\lambda}_{N1} = 0.$$

The function $\bar{F}_{N1}$ is the solution of the following boundary-value problem:

$$L_2(\bar{F}_{N1}) = -\frac{\beta}{2} N_1(\hat{w}_N, \hat{w}_N), \tag{4.11}$$

$$\frac{1}{x}\frac{\partial \bar{F}_{N1}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \bar{F}_{N1}}{\partial \theta^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \bar{F}_{N1}}{\partial \theta}\right) = 0, \quad \text{for } x = c, 1, \tag{4.12}$$

$$\int_0^{2\pi} \Sigma_1(\bar{F}_{N1})|_{x=c}\, d\theta = \int_0^{2\pi} \Sigma_2(\bar{F}_{N1})|_{x=c} \cos\theta\, d\theta = \int_0^{2\pi} \Sigma_2(\bar{F}_{N1})|_{x=c} \sin\theta\, d\theta = 0, \tag{4.13}$$

$$\bar{F}_{N1}(1, 0) = \bar{F}_{N1}(1, \pi) = \frac{\partial \bar{F}_{N1}}{\partial \theta}(1, 0) = 0; \tag{4.14}$$

and $\bar{W}_{N2}$ and $\bar{\lambda}_{N2}$ satisfy the following equation and conditions:

$$L_1(\bar{W}_{N2}) - \lambda^* \left[\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\partial^2 \bar{W}_{N2}}{\partial x^2} + \frac{1}{x}\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\left(\frac{\partial \bar{W}_{N2}}{\partial x} + \frac{1}{x}\frac{\partial^2 \bar{W}_{N2}}{\partial \theta^2}\right)\right]$$

$$= \bar{\lambda}_{N2}\left[\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\partial^2 \hat{w}_N}{\partial x^2} + \frac{1}{x}\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\left(\frac{\partial \hat{w}_N}{\partial x} + \frac{1}{x}\frac{\partial^2 \hat{w}_N}{\partial \theta^2}\right)\right] + N_1(\bar{F}_{N1}, \hat{w}_N), \tag{4.15}$$

$$\bar{W}_{N2} = \frac{\partial \bar{W}_{N2}}{\partial \theta} = 0, \quad \text{for } x = 1, \tag{4.16$_a$}$$

$$\begin{aligned} &\frac{\partial^2 \bar{W}_{N2}}{\partial x^2} + \nu_\theta\left(\frac{1}{x}\frac{\partial \bar{W}_{N2}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \bar{W}_{N2}}{\partial \theta^2}\right) = 0, \qquad \text{for } x = c, \\ &\frac{\partial^3 \bar{W}_{N2}}{\partial x^3} + \frac{1}{x}\frac{\partial^2 \bar{W}_{N2}}{\partial x^2} + \frac{2\alpha - \nu_\theta}{x^2}\left(\frac{\partial^3 \bar{W}_{N2}}{\partial x\,\partial \theta^2} - \frac{1}{x}\frac{\partial^2 \bar{W}_{N2}}{\partial \theta^2}\right) \\ &\quad - \frac{\beta}{x^2}\left(\frac{\partial \bar{W}_{N2}}{\partial x} + \frac{1}{x}\frac{\partial^2 \bar{W}_{N2}}{\partial \theta^2}\right) = 0, \end{aligned} \tag{4.16$_b$}$$

$$\int_0^{2\pi}\int_c^1 \bar{W}_{N2}(x, \theta)\,\hat{w}_N(x, \theta)\,\mathrm{d}x\,\mathrm{d}\theta = 0. \tag{4.17}$$

By substituting the eigenfunction $\hat{w}_N(x, \theta) = \hat{w}_N(x)\cos N\theta$ of (3.7) and (3.8) into the right-hand side of (4.11), we have

$$-\tfrac{1}{2}\beta N_1(\hat{w}_N, \hat{w}_N) \equiv f_1(x) + f_2(x)\cos 2N\theta,$$

in which $f_1(x)$ and $f_2(x)$ are two known functions. We therefore may set

$$\begin{aligned} \bar{F}_{N1}(x, \theta) &= \bar{\psi}_1(x) + \bar{\psi}_2(x)\cos 2N\theta, \\ \psi_1(x) &\equiv \frac{\mathrm{d}\bar{\psi}_1}{\mathrm{d}x}, \quad \psi_2(x) \equiv \bar{\psi}_2(x). \end{aligned} \tag{4.18}$$

By substituting (4.18) into (4.11)–(4.14), we obtain the two boundary-value problems for $\psi_1(x)$ and $\psi_2(x)$

$$\text{(I)} \quad \begin{cases} \dfrac{\mathrm{d}^3\psi_1}{\mathrm{d}x^3} + \dfrac{2}{x}\dfrac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} - \dfrac{\beta}{x^2}\dfrac{\mathrm{d}\psi_1}{\mathrm{d}x} + \dfrac{\beta}{x^3}\psi_1 = f_1(x), \\ \psi_1(1) = \psi_1(c) = 0, \\ \left[\dfrac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} + \dfrac{1}{x}\dfrac{\mathrm{d}\psi_1}{\mathrm{d}x} - \dfrac{\beta}{x^2}\psi_1\right]\Bigg|_{x=c} = 0; \end{cases}$$

and

$$
\text{(II)}\left\{\begin{aligned}
&\frac{\mathrm{d}^4\psi_2}{\mathrm{d}x^4}+\frac{2}{x}\frac{\mathrm{d}^3\psi_2}{\mathrm{d}x^3}-\frac{\beta-8\delta N^2}{x^2}\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2}+\frac{\beta-8\delta N^2}{x^3}\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\\
&\qquad+\frac{16\beta N^4-8N^2(\beta-\delta)}{x^4}\psi_2=f_2(x),\\
&\psi_2=\frac{\mathrm{d}\psi_2}{\mathrm{d}x}=0,\quad \text{for } x=c,\ 1.
\end{aligned}\right.
$$

It is not difficult to obtain that the solutions $\psi_1(x)$ and $\psi_2(x)$ to the problems (I) and (II) are

$$
\begin{aligned}
\psi_1(x) &= \left(A_1\int_c^x t^{2-\sqrt{\beta}}f_1(t)\,\mathrm{d}t+C_1\right)x^{\sqrt{\beta}}+\left(A_2\int_c^x t^2 f_1(t)\,\mathrm{d}t+C_2\right)x\\
&\quad+\left(A_3\int_c^x t^{2+\sqrt{\beta}}f_1(t)\,\mathrm{d}t+C_3\right)x^{-\sqrt{\beta}},\\
\psi_2(x) &= \left(\bar{A}_1\int_c^x t^{7-m_1}f_2(t)\,\mathrm{d}t+\bar{C}_1\right)x^{m_1}+\left(\bar{A}_2\int_c^x t^{7-m_2}f_2(t)\,\mathrm{d}t+\bar{C}_2\right)x^{m_2}\\
&\quad+\left(\bar{A}_3\int_c^x t^{7-m_3}f_2(t)\,\mathrm{d}t+\bar{C}_3\right)x^{m_3}+\left(\bar{A}_4\int_c^x t^{7-m_4}f_2(t)\,\mathrm{d}t+\bar{C}_4\right)x^{m_4},
\end{aligned}
$$

in which A_i, C_i and $\bar{A}_i$, $\bar{C}_i$ as well as m_i are all constants. Substituting $\psi_i(x)$ into (4.18) and then substituting $\bar{F}_{N1}$, $\bar{F}^0$ and $\hat{w}_N$ into the right-hand side of (4.15), we may calculate the result which possesses the form $f_3(x, \bar{\lambda}_{N2})\cos N\theta + f_4(x)\cos 3N\theta$, where the functions $f_3(x, \bar{\lambda}_{N2})$ and $f_4(x)$ are known. We may therefore assume that

$$
\bar{W}_{N2}(x,\theta) = \phi_1(x)\cos N\theta+\phi_2(x)\cos 3N\theta. \tag{4.19}
$$

By substituting $\bar{W}_{N2}(x, \theta)$ into (4.15)–(4.17), we obtain the equations and boundary conditions for $\phi_1(x)$ and $\phi_2(x)$ as follows:

$$
\text{(III)}\left\{\begin{aligned}
&\frac{\mathrm{d}^4\phi_1}{\mathrm{d}x^4}+\frac{2}{x}\frac{\mathrm{d}^3\phi_1}{\mathrm{d}x^3}-\frac{\beta+2\alpha N^2}{x^2}\left(\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2}-\frac{1}{x}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\right)+\frac{\beta N^4-2(\alpha+\beta)N^2}{x^4}\phi_1\\
&\quad-\lambda^*\left(\frac{1}{x}\frac{\mathrm{d}\bar{F}^0}{\mathrm{d}x}\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}-\frac{N^2}{x^2}\frac{\mathrm{d}^2\bar{F}^0}{\mathrm{d}x^2}\phi_1\right)=f_3(x,\bar{\lambda}_{N2}),\\
&\phi_1=\frac{\mathrm{d}\phi_1}{\mathrm{d}x}=0,\quad\text{for } x=1,\\
&\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2}+\nu_0\left(\frac{1}{x}\frac{\mathrm{d}\phi_1}{\mathrm{d}x}-\frac{N^2}{x^2}\phi_1\right)=0,\quad\text{for } x=c,\\
&\frac{\mathrm{d}^3\phi_1}{\mathrm{d}x^3}+\frac{1}{x}\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2}+\frac{2\alpha-\nu_\theta}{x^2}\left(\frac{N^2}{x}\phi_1-N^2\frac{\mathrm{d}\phi_1}{\mathrm{d}x}\right)-\frac{\beta}{x^2}\left(\frac{\mathrm{d}\phi_1}{\mathrm{d}x}-\frac{N^2}{x}\phi_1\right)=0,\\
&\int_c^1\phi_1(x)\,\hat{w}_N(x)\,\mathrm{d}x=0;
\end{aligned}\right.
$$

and

$$
\text{(IV)}\begin{cases}
\dfrac{d^4\phi_2}{dx^4} + \dfrac{2}{x}\dfrac{d^3\phi_2}{dx^3} - \dfrac{8 + 18\alpha N^2}{x^2}\left(\dfrac{d^2\phi_2}{dx^2} - \dfrac{1}{x}\dfrac{d\phi_2}{dx}\right) + \dfrac{81\beta N^4 - 18(\alpha + \beta)N^2}{x^4}\phi_2 \\
\quad - \lambda^*\left(\dfrac{1}{x}\dfrac{d\bar{F}^0}{dx}\dfrac{d^2\phi_2}{dx^2} + \dfrac{1}{x}\dfrac{d^2\bar{F}^0}{dx^2}\dfrac{d\phi_2}{dx} - \dfrac{9N^2}{x^2}\dfrac{d^2\bar{F}^0}{dx^2}\phi_2\right) = f_4(x), \\
\phi_2 = \dfrac{d\phi_2}{dx} = 0, \quad \text{for } x = 1, \\
\dfrac{d^2\phi_2}{dx^2} + \nu_0\left(\dfrac{1}{x}\dfrac{d\phi_2}{dx} - \dfrac{N^2}{x^2}\phi_2\right) = 0, \quad \text{for } x = c, \\
\dfrac{d^3\phi_2}{dx^3} + \dfrac{1}{x}\dfrac{d^2\phi_2}{dx^2} + \dfrac{9(2\alpha - \nu_\theta)N^2 - \beta}{x^2}\dfrac{d\phi_2}{dx} - \dfrac{9(2\alpha - \nu_\theta - \beta)}{x^3}N^2\phi_2 = 0.
\end{cases}
$$

By using finite-difference methods, we can obtain numerical solutions to the problems (I)–(IV). Thus, the bifurcation solutions of the trivial solution of (EP) near $\lambda = \lambda^*$ have the asymptotic formulae

$$
\begin{aligned}
W_N(x, \theta) &= \varepsilon\hat{w}_N(x, \theta) + \varepsilon^3\bar{W}_{N2}(x, \theta) + O(\varepsilon^4) \\
&= \varepsilon\hat{w}_N(x)\cos N\theta + \varepsilon^3(\phi_1(x)\cos N\theta + \phi_2(x)\cos 3N\theta) + O(\varepsilon^4), \\
F_N(x, \theta) &= F^0(x) + \varepsilon^2\bar{F}_{N1}(x, \theta) + O(\varepsilon^4) \\
&= F^0(x) + \varepsilon^2(\bar{\psi}_1(x) + \bar{\psi}_2(x)\cos 2N\theta) + O(\varepsilon^4), \\
\lambda_N &= \lambda^* + \varepsilon^2\bar{\lambda}_{N2} + O(\varepsilon^3).
\end{aligned} \tag{4.20}
$$

The stress is given in terms of the non-dimensional variables by

$$
\begin{aligned}
\Sigma_x &= \frac{1}{x}\frac{\partial F_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F_N}{\partial \theta^2} \pm \frac{1}{1 - \nu_r\nu_\theta}\left[\frac{\partial^2 W_N}{\partial x^2} + \nu_\theta\left(\frac{1}{x}\frac{\partial W_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 W_N}{\partial \theta^2}\right)\right], \\
\Sigma_\theta &= \frac{\partial^2 F_N}{\partial x^2} \pm \frac{1}{1 - \nu_r\nu_\theta}\left(\frac{1}{x}\frac{\partial W_N}{\partial x} + \frac{1}{x^2}\frac{\partial^2 W_N}{\partial \theta^2} + \nu_r\frac{\partial^2 W_N}{\partial x^2}\right), \\
\Sigma_{x\theta} &= -\left(\frac{1}{x}\frac{\partial^2 F_N}{\partial x\partial\theta} - \frac{1}{x^2}\frac{\partial F_N}{\partial\theta}\right) \pm \frac{2(\alpha - \nu_\theta)}{1 - \nu_r\nu_\theta}\left(\frac{1}{x}\frac{\partial^2 W_N}{\partial x\partial\theta} - \frac{1}{x^2}\frac{\partial^2 W_N}{\partial\theta^2}\right).
\end{aligned} \tag{4.21}
$$

The curves of the deflection and stress for various values of the material parameter β are shown in Figs. 1–3 for particular values of N, θ and c.

We can see that, from Tables 1, 2 and Figs. 1–3, the effects of the parameters β and c on λ^* and N as well as on the buckled states are all substantial. The effect of β on the deflection decreases gradually with increasing c and the effect of N on the stress is also considerable. We also see that the stress in the buckled plate consists mainly of the membrane forces but that the shear stress is no longer zero.

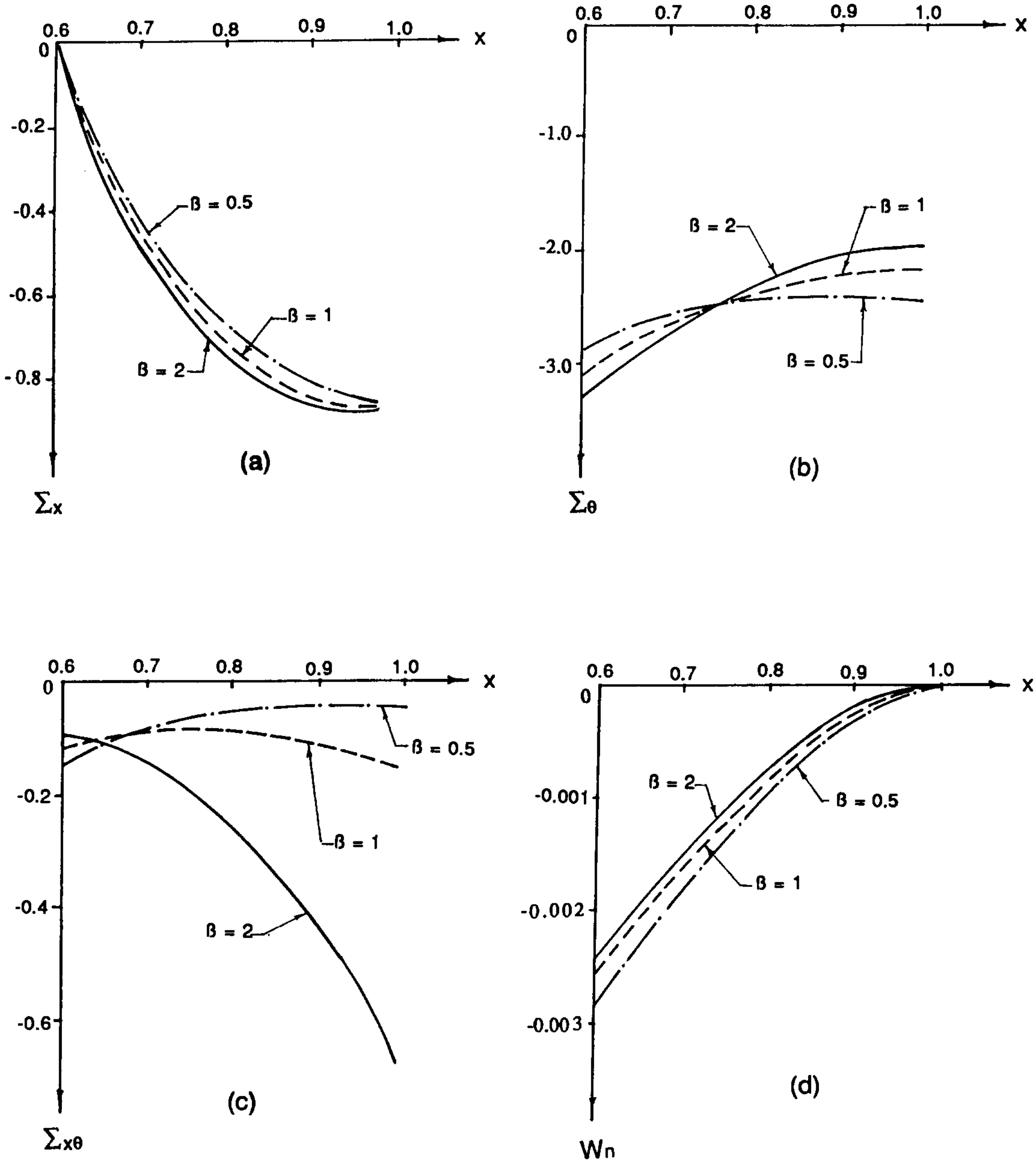

Fig. 1. Curves of deflection and stress (for $N = 2$, $\theta = 30°$, $c = 0.6$ and $\varepsilon = 0.1$).

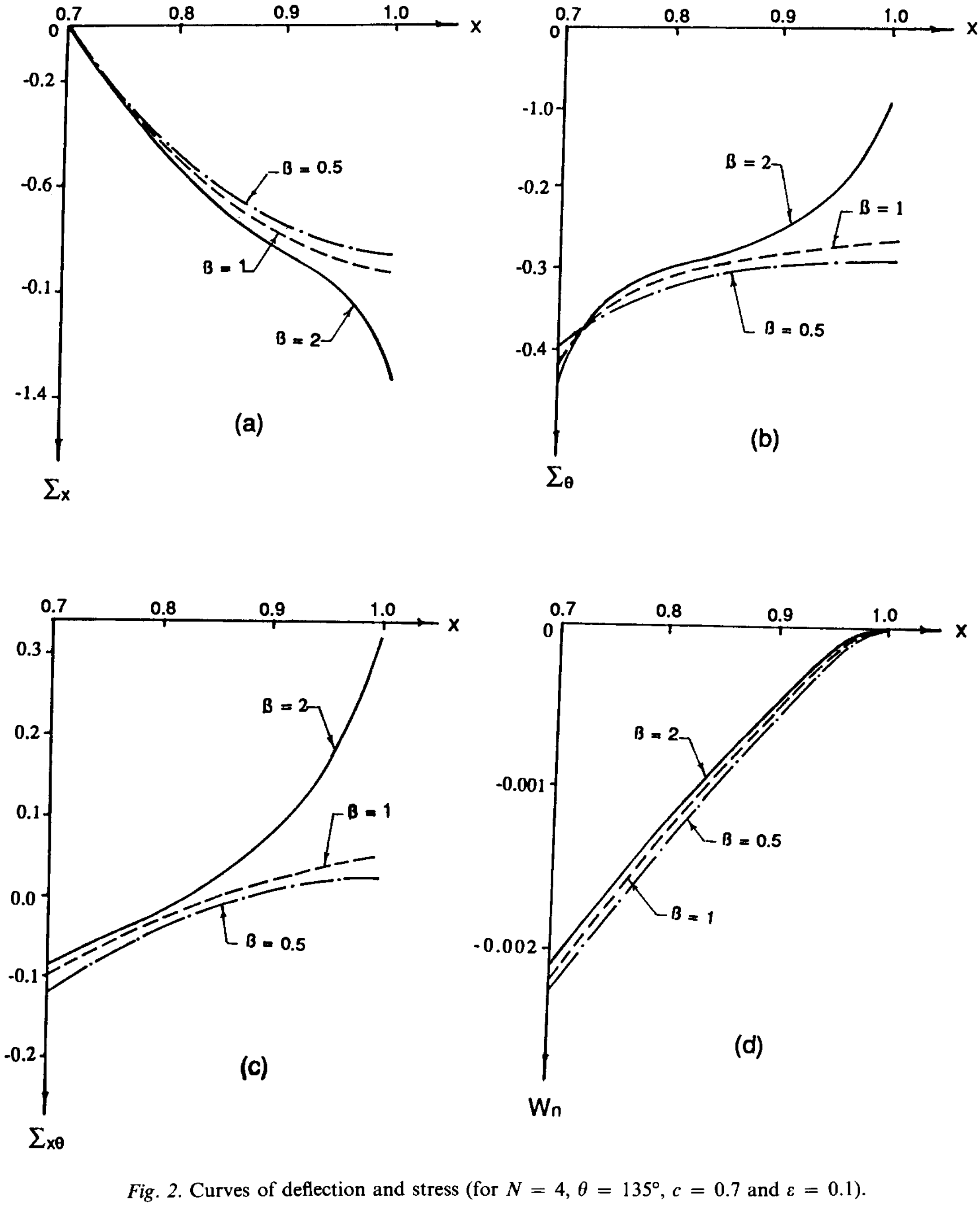

Fig. 2. Curves of deflection and stress (for $N = 4$, $\theta = 135°$, $c = 0.7$ and $\varepsilon = 0.1$).

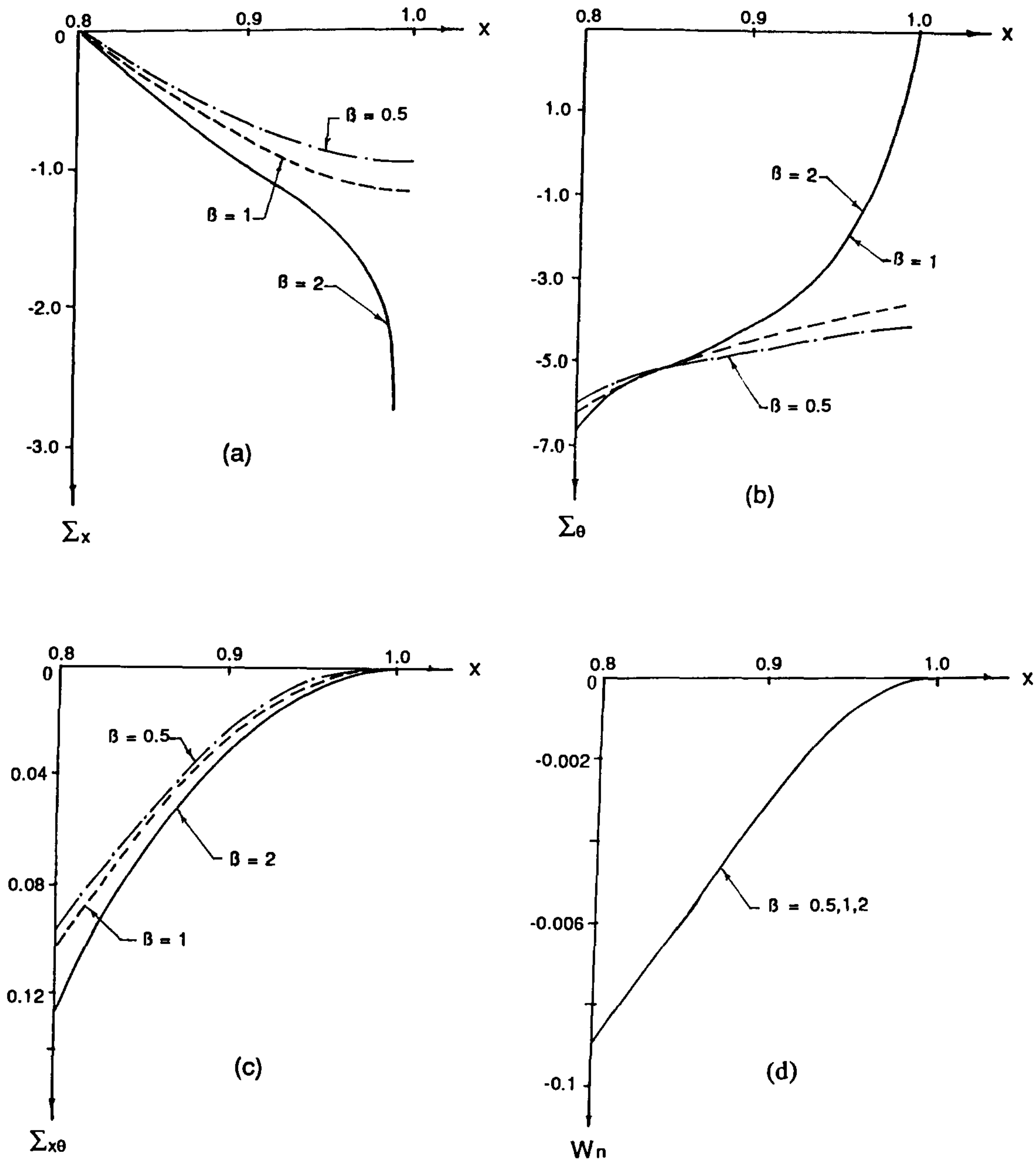

Fig. 2. Curves of deflection and stress (for $N = 7$, $\theta = 0°$, $c = 0.8$ and $\varepsilon = 0.1$).

Acknowledgement

The work described in this paper was supported by the National Natural Science Foundation of China and a Sino-British Friendship Scholarship.

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